Fractional Fokker-Planck equation, solution, and application

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Recently, Metzler *et al.* [Phys. Rev. Lett. **82**, 3563 (1999)], introduced a fractional Fokker-Planck equation (FFPE) describing a subdiffusive behavior of a particle under the combined influence of external nonlinear force field, and a Boltzmann thermal heat bath. In this paper we present the solution of the FFPE in terms of an integral transformation. The transformation maps the solution of ordinary Fokker-Planck equation onto the solution of the FFPE, and is based on Lévy's generalized central limit theorem. The meaning of the transformation is explained based on the known asymptotic solution of the continuous time random walk (CTRW). We investigate in detail (i) a force-free particle, (ii) a particle in a uniform field, and (iii) a particle in a harmonic field. We also find an exact solution of the CTRW, and compare the CTRW result with the corresponding solution of the FFPE. The relation between the fractional first passage time problem in an external nonlinear field and the corresponding integer first passage time is given. An example of the one-dimensional fractional first passage time in an external linear field is investigated in detail. The FFPE is shown to be compatible with the Scher-Montroll approach for dispersive transport, and thus is applicable in a large variety of disordered systems. The simple FFPE approach can be used as a practical tool for a phenomenological description of certain types of complicated transport phenomena.

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I. INTRODUCTION

Fractional kinetic equations were introduced to describe anomalous types of relaxation and diffusion processes: for example, relaxation processes in viscoelastic media [1,2], protein dynamics [3], and diffusion processes found in chaotic Hamiltonian systems [4]. Schneider and Wyss [5] introduced a fractional diffusion equation describing a subdiffusive process investigated in Ref. [6], where $\langle r^2\rangle \sim t^\alpha$ and $0<\alpha<1$. The fractional diffusion equation describes the asymptotic behavior of the continuous time random walk [6–9], which in turn is known to describe different types of anomalous transport [10,11]. Recently, a fractional Fokker-Planck equation (FFPE) describing such an anomalous subdiffusive behavior in an external nonlinear field $F(x)$, and close to thermal equilibrium, was investigated [12], and when $F(x)=0$ the equation coincides with the fractional diffusion equation. In Refs. [13,14] the FFPE was derived from a generalized continuous time random walk (CTRW), which includes space dependent jump probabilities which are the result of an external field $F(x)$.

The main purpose of this paper is to present a simple method of solving the FFPE. The solution is based on an integral transformation which maps a Gaussian type of diffusion onto fractional diffusion. The transformation was investigated by Bouchaud and Georges [15] and Klafter and Zumofen [16] in the context of the CTRW. While Saichev and Zaslavsky (Ref. [17] and later Ref. [18]) considered the transformation in the context of fractional kinetic equations, here we generalize these results for the kinetics described by the FFPE in Ref. [12].

The integral transformation we investigate maps a solution of the ordinary Fokker-Planck equation $P_1(x,t)$, onto the corresponding solution of the FFPE $P_\alpha(x,t)$, according to

$$P_\alpha(x,t)=\int_0^\infty n(s,t)P_1(x,s)ds, \tag{1}$$

where $0<\alpha<1$, and

$$n(s,t)=\frac{d}{ds}\left[1-L_\alpha\left(\frac{t}{s^{1/\alpha}}\right)\right] \tag{2}$$

denotes the *inverse* one sided Lévy stable density [i.e., $L_\alpha(x)$ is the one sided Lévy stable distribution [19,20]]. In Eq. (1), the probability density function (PDF) $P_\alpha(x,t)$ has the same initial and boundary conditions as the corresponding $P_1(x,t)$. We call Eq. (1) an inverse Lévy transform. For example consider the force free case with free boundary conditions and initial conditions concentrated on the origin. The solution of the FFPE $P_\alpha(x,t)$ is found by transforming $P_1(x,s)$, and the transformed function is the well known Gaussian solution of the integer diffusion equation. Transformations similar to Eq. (1) hold for dimensions higher than 1. In what follows, we suppress the subscript α in $P_\alpha(x,t)$.

In Sec. II we give definitions, and briefly recall other fractional approaches related to the FFPE under investigation. A solution of the problem is discussed in Sec. III. Section IV explains the meaning of the transformation based on the CTRW.

We then discuss an application. Scher and Montroll (SM) [21] modeled transport in a disordered medium based on the CTRW. Using SM predictions, one can explain and fit a large number of experimental results. For example, recent experiments in organic photorefractive glasses [22], nanocrystalline T_iO_2 electrodes [23], and conjugated polymer system poly *p*-phenylene [24] indicate that the Scher-Montroll results are truly universal. Can one use the FFPE to model the SM type of transport? Since transport processes

(e.g., time of flight measurements) are sensitive to boundary conditions, it is not obvious that the FFPE and CTRW predict the same result. Generally a CTRW with boundary conditions cannot be assumed to be equivalent to the corresponding FFPE [13]. Here we show that the FFPE is compatible with the SM predictions. The FFPE approach is a straightforward approach which simplifies the much more complex CTRW. Methods of solutions of standard Fokker-Planck equation can be used to solve the FFPE, and in some cases analytical results can be found [12]. These simplifications are the main reason why a stochastic fractional approach was introduced.

II. FRACTIONAL FOKKER-PLANCK EQUATION

Let $P(x,t)\geqslant 0$ be a normalized probability density,

$$\int_{-\infty}^{\infty} P(x,t)dx=1, \tag{3}$$

to find a particle on x at time t. A Gaussian Markovian type of diffusion, in an external field $F(x)$, and close to thermal equilibrium, was modeled many times based on the linear Smolochowski Fokker-Planck (FP) equation [25,26]

$$\frac{\partial P(x,t)}{\partial t}=K_1\tilde{L}_{\rm FP}P(x,t), \tag{4}$$

with the operator

$$\tilde{L}_{\rm FP}=-\frac{\partial}{\partial x}\frac{F(x)}{T}+\frac{\partial^2}{\partial x^2}, \tag{5}$$

where K_1 and T are the diffusion coefficient and temperature, respectively. We consider a generalization of Eq. (4) based on *fractional Riemann-Liouville integration*. Let us rewrite Eq. (4) in an integral form,

$$P(x,t)-\delta(x-x_0)=K_1\,{}_0D_t^{-1}\tilde{L}_{\rm FP}P_1(x,t), \tag{6}$$

where $\delta(x-x_0)$ is the initial condition, and we shall assume free boundary conditions. Replacing the integer integral operator ${}_0D_t^{-1}$ in Eq. (6) with a fractional Riemann-Liouville integral operator [27,28]

$${}_0D_t^{-\alpha}Z(t)\equiv\frac{1}{\Gamma(\alpha)}\int_0^t\frac{Z(t')}{(t-t')^{1-\alpha}}dt'. \tag{7}$$

We find

$$P_\alpha(x,t)-\delta(x-x_0)=K_\alpha\,{}_0D_t^{-\alpha}\tilde{L}_{\rm FP}P_\alpha(x,t), \tag{8}$$

where K_α is a generalized diffusion coefficient. An ordinary time differentiation of the fractional integral [Eq. (8)] yields the FFPE [12]:

$$\frac{\partial P(x,t)}{\partial t}=K_\alpha\,{}_0D_t^{1-\alpha}\tilde{L}_{\rm FP}P(x,t). \tag{9}$$

It is easy to show that $P(x,t)$ in Eq. (9) is normalized; in Sec. III we show that the solution is also non-negative. As mentioned in Sec. I, the FFPE (9) was derived from a generalized CTRW in Ref. [13]. When $F(x)=0$, the equation coincides with the Schneider-Wyss fractional diffusion equation [5]. Later we shall use the Laplace transform of Eq. (9) or (8) [29],

$$u\hat{P}(x,u)-\delta(x-x_0)=K_\alpha u^{1-\alpha}\tilde{L}_{\rm FP}\hat{P}(x,u), \tag{10}$$

where the Laplace transform is defined:

$$\hat{P}(x,u)=\int_0^\infty P(x,t)e^{-ut}dt. \tag{11}$$

Let us briefly recall [12] known properties of the FFPE (9). (i) In the presence of an external time independent binding external field, the stationary solution is the Boltzmann distribution. (ii) Generalized Einstein relations are satisfied consistently with linear response theory [30]. (iii) Relaxation of single modes follows Mittag-Leffler relaxation (related for example to Cole-Cole relaxation [31] and to work in [1–3]). (iv) In the limit $\alpha\to 1$, the standard Smolochowski Fokker-Planck equation (4) is recovered.

In Ref. [32] a different fractional Fokker-Planck equation, based on *fractional time derivatives*, was investigated. Following Ref. [32] we replace the ordinary time derivative ∂_t in the Fokker-Planck equation with a fractional time derivative

$${}_0D_t^{\alpha}Z(t)\equiv\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t\frac{Z(t')}{(t-t')^{\alpha}}dt', \tag{12}$$

and then Eq. (4) is replaced with

$${}_0D_t^{\alpha}P(x,t)=K_\alpha\tilde{L}_{\rm FP}P(x,t), \tag{13}$$

where $0<\alpha<1$. Jumarie [32] noted that such a fractional Fokker-Planck equation is not generally valid, since the PDF $P(x,t)$ is not a normalized non-negative function (the discussion on this point in Ref. [32] is not very clear; see further discussion in Ref. [33] and briefly in Ref. [12]). Interestingly, Arkhincheev [34] considered a random walk on a comb structure for which the number of particles is not conserved [i.e., Eq. (3) is not valid], and showed that Eq. (13) [with $F(x)=0$ and $\alpha=1/2$] describes such a process.

It is worth mentioning that other types of fractional Fokker-Planck equations based on space fractional derivatives were investigated in Ref. [4,35–39] also (see Refs. [40–43] for related works and discussion). Space fractional equations were used to describe Lévy flights (e.g., in chaotic Hamiltonian systems [4]), but in contrast we are considering subdiffusive behavior. In Ref. [18] a fractional Kramers equation approach describing superdiffusion was introduced.

III. FFPE SOLUTION

In order to find solution of FFPE (9), we introduce the ansatz

$$P(x,t)=\int_0^\infty n(s,t)P_1(x,s)ds, \tag{14}$$

where

$$\frac{\partial P_1(x,s)}{\partial s}=\tilde{L}_{\rm FP}P_1(x,s). \tag{15}$$

$P_1(x,s)$ in Eq. (15) is a normalized solution of the ordinary Fokker-Planck equation with initial conditions $P_1(x,s=0)=\delta(x-x_0)$ and free boundary conditions. We shall soon prove that Eq. (1) is valid, namely,

$$n(s,t)=\frac{1}{\alpha}\frac{t}{s^{1+1/\alpha}}l_\alpha\left(\frac{t}{s^{1/\alpha}}\right), \tag{16}$$

where $l_\alpha(z)$ in Eq. (16) is a one sided Lévy stable probability density whose Laplace $z\to u$ transform is

$$\hat{l}_\alpha(u)=\exp(-u^\alpha). \tag{17}$$

We then show that Eq. (14) is identical to the solution found in Ref. [12], thus justifying our initial ansatz. For $F(x)=0$, and within the context of fractional kinetic equations, Eq. (14) was first investigated in Ref. [17]. In Eqs. (14) and (16) we used $K_\alpha=1$; later we restore physical units.

We use the Laplace transform of Eq. (14), and the normalization condition of $P(x,t)$ to show

$$\int_0^\infty \hat{n}(s,u)ds=1/u. \tag{18}$$

From Eq. (18) we see that $n(s,t)$ is normalized according to $\int_0^\infty n(s,t)ds=1$. Inserting Eq. (14) into Eq. (10), we find

$$u\int_0^\infty \hat{n}(s,u)P_1(x,s)ds-\delta(x-x_0)$$
$$=u^{1-\alpha}\int_0^\infty \hat{n}(s,u)\tilde{L}_{\rm FP}P_1(x,s)ds. \tag{19}$$

Integrating by parts using Eq. (15), we find

$$u\int_0^\infty \hat{n}(s,u)P_1(x,s)ds-\delta(x-x_0)$$
$$=u^{1-\alpha}[\hat{n}(\infty,u)P_1(x,s=\infty)-\hat{n}(0,u)P_1(x,s=0)]$$
$$-u^{1-\alpha}\int_0^\infty\left[\frac{\partial}{\partial s}\hat{n}(s,u)\right]P_1(x,s)ds. \tag{20}$$

From Eq. (18), $\hat{n}(\infty,u)=0$, and $P_1(x,s=\infty)$ in Eq. (20) is the stationary solution of the standard Fokker-Planck equation, namely, the Boltzmann distribution (i.e., assuming binding potential), and since $P_1(x,s=0)=\delta(x-x_0)$ we may rewrite Eq. (20):

$$\int_0^\infty\left\{u\hat{n}(s,u)+u^{1-\alpha}\left[\frac{\partial}{\partial s}\hat{n}(s,u)\right]\right\}P_1(x,s)ds$$
$$=[1-u^{1-\alpha}\hat{n}(0,u)]\delta(x-x_0). \tag{21}$$

Equation (21) is solved once both of its sides are equal to zero; therefore, two conditions must be satisfied:

$$\hat{n}(0,u)=u^{\alpha-1} \tag{22}$$

and

$$\frac{\partial}{\partial s}\hat{n}(s,u)=-u^\alpha\hat{n}(s,u). \tag{23}$$

The solution of Eq. (23), with initial condition (22), is

$$\hat{n}(s,u)=u^{\alpha-1}\exp(-u^\alpha s). \tag{24}$$

The inverse Laplace transform of $\hat{n}(s,u)$ is $n(s,t)$ [Eq. (16)].

Let us now show that solution (14) is identical to the solution found in Ref. [12]. We rewrite $P_1(x,s)$ using a well known eigenfunction expansion [26]

$$P_1(x,s)=e^{\Phi(x)/2-\Phi(x_0)/2}\sum_n \psi_n(x)\psi_n(x_0)e^{-\lambda_n s}. \tag{25}$$

Here the functions $\psi_n(x)=e^{\Phi(x)/2}\phi_n(x)$ are related to eigenfunctions of the Fokker-Planck operator $\tilde{L}_{\rm FP}$, $\phi_n(x)$ via the scaled potential $V(x)/T$ and $F(x)=-V'(x)$, while λ_n are the corresponding eigenvalues [26]. We now insert Eq. (25) into the Laplace transform of Eq. (14) using Eq. (24):

$$\hat{P}(x,u)=\int_0^\infty u^{\alpha-1}e^{-su^\alpha}e^{\Phi(x)/2-\Phi(x_0)/2}$$
$$\times\sum_n \psi_n(x)\psi_n(x_0)e^{-\lambda_n s}ds. \tag{26}$$

We change the order of integration and summation, and use the identity

$$u^{\alpha-1}\int_0^\infty e^{-su^\alpha-s\lambda_n}ds=\frac{u^{\alpha-1}}{u^\alpha+\lambda_n}; \tag{27}$$

applying the inverse Laplace transform $u\to t$, we find

$$P(x,t)=e^{\Phi(x)/2-\Phi(x_0)/2}\sum_n \psi_n(x)\psi_n(x_0)E_\alpha(-\lambda_n t^\alpha). \tag{28}$$

where $E_\alpha(z)$ is the Mittag-Leffler function. Equation (28) was found in Ref. [12], based on the method of separation of variables.

Remark 1. $n(s,t)$ is a non-negative PDF normalized according to $\int_0^\infty n(s,t)ds=1$. It is easy to understand that solution (14) is normalized and non-negative, since both $P_1(x,s)$ and $n(s,t)$ are PDF's.

Remark 2. Properties of $n(s,t)$ can be found based on known properties of $l_\alpha(z)$ discussed in Appendix A (also see Ref. [17]). For $\alpha=1$, $n(s,t)=\delta(s-t)$, as expected.

Remark 3. Deriving Eq. (14), we assumed free boundary conditions. One can generalize our results for other types of boundary conditions—for example, an absorbing boundary condition which is important for the analysis of anomalous current in time of flight experiments, which we discuss in Sec V. Generalization of our result for dimension $d>1$ is straightforward.

Remark 4. We now restore physical units, using Eq. (16),

$$P(x,t)=\frac{1}{\alpha}\left(\frac{K_\alpha}{K_1}\right)^{1/\alpha} t\int_0^\infty \frac{1}{t'^{1+1/\alpha}} l_\alpha\left(\frac{K_\alpha^{1/\alpha}t}{K_1^{1/\alpha}t'^{1/\alpha}}\right)P_1(x,t')dt', \tag{29}$$

and $P_1(x,t')$ is the corresponding solution of the ordinary Fokker-Planck equation in the time domain. The solution $P(x,t)$ does not depend on the arbitrary choice of K_1, while obviously $P_1(x,t')$ does.

Remark 5. Transformations (1), (14), and (29) have the same meaning. We have suggested calling this transformation an inverse Lévy transform. The meaning of the transformation can be understood based on the CTRW discussed in Sec. IV. The transformation, for the force free case, appeared previously in Refs. [15–18] (also see Ref. [44] for related work).

Remark 6. It might be interesting to see if dynamics of the FFPE are compatible with Hilfer's theory of fractional dynamics [45] (i.e., see Eq. 3.11 in Ref. [45]).

A. Example 0, force free fractional diffusion

The aim now is to show that the solution found in Sec. II is identical to the known solution of a fractional diffusion equation in d dimensions [5]

$$\frac{\partial P(r,t)}{\partial t}={}_0D_t^{1-\alpha}\nabla^2 P(r,t). \tag{30}$$

It is easy to generalize Eq. (14) to dimension $d>1$. For the force free case we then find

$$P(r,t)=\int_0^\infty n(s,t)\frac{1}{(4\pi s)^{d/2}}\exp\left(-\frac{r^2}{4s}\right)ds, \tag{31}$$

where we used the well known Gaussian solution of the ordinary diffusion equation (i.e., assuming initial conditions concentrated on the origin). Applying the Laplace transform with respect to time t,

$$\hat{P}(r,u)=\frac{u^{\alpha-1}}{(2\pi)^{d/2}}\left(\frac{r}{u^{\alpha/2}}\right)^{1-d/2}K_{d/2-1}(ru^{\alpha/2}), \tag{32}$$

where $K_{d/2-1}$ denotes the Bessel function of the second kind. Schneider and Wyss [5], Laplace inverted Eq. (32) using the Mellin transform, and found a solution in terms of the Fox H function:

$$P(r,t)=\alpha^{-1}\pi^{-d/2}r^{-d}H_{12}^{20}\left(2^{-2/\alpha}r^{2/\alpha}t^{-1}\middle|\begin{matrix}(1,1)\\(d/2,1/\alpha),(1,1/\alpha)\end{matrix}\right). \tag{33}$$

For $d=1$ the solution can be represented in terms of stable densities [5]. The asymptotic $\xi\equiv r^2/t^\alpha\gg 1$ behavior is [5]

$$P(r,t)\sim\kappa^\alpha r^{-d}\xi^{d/2(2-\alpha)}\exp(-\lambda_1\xi^{1/(2-\alpha)}), \tag{34}$$

where

$$\kappa^\alpha=\pi^{-d/2}2^{-d(2-\alpha)}(2-\alpha)^{-1/2}\alpha^{[\alpha(d+1)/2-1]/(2-\alpha)]}$$

and

$$\lambda_1=(2-\alpha)\alpha^{\alpha/(2-\alpha)}2^{-2/(2-\alpha)}.$$

The behavior of $P(r,t)$ for $\xi\ll 1$ and $d=3$ is found in Appendix B, based on the series expansion of the H function [46,47]:

$$P(r,t)=\frac{1}{4\pi t^{3\alpha/2}\xi^{1/2}}\sum_{n=0}^\infty\frac{(-1)^n\xi^{n/2}}{n!\Gamma[1-\alpha(1+n/2)]}. \tag{35}$$

We see that for $d=3$, $\alpha\neq 1$, and when $r\to 0$ the normalized solution diverges like $P(r,t)\sim 1/r$ as pointed out in Ref. [48]. The behavior in dimension $d=1$ and 2 is briefly discussed in the Appendixes.

While the solution in terms of the Fox H function is clearly a step forward, one cannot in general find tables or numerical packages with which explicit values of the solution can be found (i.e., for $d>1$). Therefore a numerical solution using the integral transformation [Eq. (31)] is an important tool. Here we check that such a numerical approach works well for $\alpha=1/2$ and $d=3$. Also, generalizations to other cases seem straightforward. To find our results we use the MATHEMATICA command NIntegrate. We are mainly concerned about whether the numerical solution works well close to the singular point $r\to 0$. In Fig. 1 we show $P(r,t)$ versus r on a semilog plot and for different times t. Close to the origin $r=0$ we observe a sharp increase of $P(r,t)$, as predicted in Eq. (35). More detailed behavior of $P(r,t)$ is presented in Fig. 2, where we show $rP(r,t)$ versus r. In Fig. 2 we also exhibit linear curves based on the asymptotic expansion [Eq. (35)], which predicts

$$rP(r,t)\sim C_1(t)-rC_2(t) \tag{36}$$

where $C_1(t)=1/(4\pi^{3/2}t^{1/2})$ and $C_2(t)=1/[4\pi\Gamma(1/4)t^{3/4}]$. Equation (36) is valid when $\xi=r^2/t^{1/2}\ll 1$, and we see that the numerical solution and the asymptotic equation (36) agree well in the limit. Finally, in Fig. 3 we show our results in a scaling form. We present $r^3P(r,t)$ versus ξ for different choices of time t observing scaling behavior of the solution. We also show the asymptotic behaviors $\xi\gg 1$ and $\xi\ll 1$, Eqs. (34) and (35), respectively.

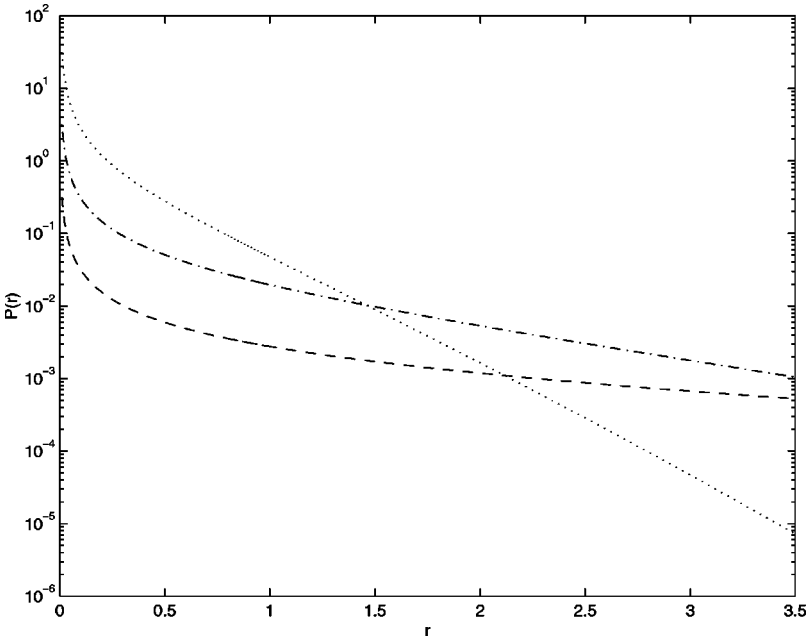

FIG. 1. $P(r,t)$ vs r on a semilog plot, and for different times $t=0.02$ (dotted line), $t=2$ (dash dotted line) and $t=200$ (dashed line), with $\alpha=1/2$ and $d=3$. Note that $P(r=0,t)=\infty$; hence the solution in the figure is cut off close to $r\rightarrow 0$. To obtain our results we used Eqs. (31) and (A7) and MATHEMATICA.

B. Example 1: biased fractional Wiener process

Consider a biased one-dimensional fractional diffusion process defined with a generalized diffusion coefficient K_α and a uniform force field $F(x)=F>0$. For this case the mean displacement grows more slowly than linearly with time, according to

$$\langle x(t)\rangle=\frac{FK_\alpha t^\alpha}{T\Gamma(1+\alpha)}. \tag{37}$$

The well known solution of the ordinary Fokker-Planck equation, with $K_1=1$, is

$$P_1(x,t')=\frac{1}{\sqrt{4\pi t'}}\exp\left[-\frac{(x-Ft'/T)^2}{4t'}\right]; \tag{38}$$

this describes a biased Wiener process. The solution $P(x,t)$ for the fractional case is found using transformation (14). In Laplace u space the solution is

$$\hat{P}(x,u)=\frac{Fu^{\alpha-1}\tau^\alpha}{T\sqrt{1+4(u\tau)^\alpha}}\times\exp\left[\frac{F[x-\sqrt{1+4(u\tau)^\alpha}|x|]}{2T}\right], \tag{39}$$

and $\tau^\alpha=T^2/(F^2K_\alpha)$. For $(\tau u)^\alpha\ll 1$, corresponding to the long time behavior of the solution $P(x,t)$, we find

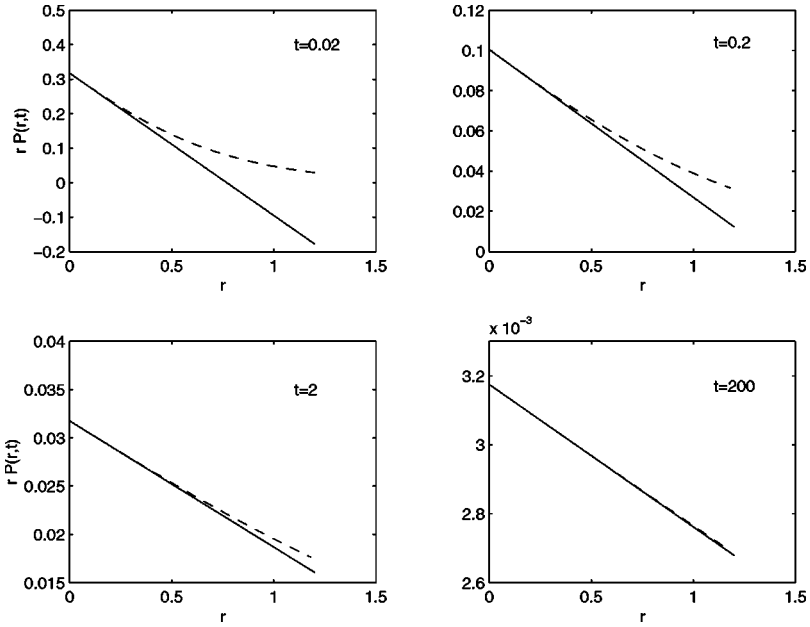

FIG. 2. Behavior of $P(r,t)$ close to the origin $r=0$ for $\alpha=1/2$ and $d=3$. We show (dashed curves) $rP(r,t)$ vs r for different times indicated in the figure. Also shown (linear curves) is the approximation Eq. (36). As expected when $\xi=r^2/t^\alpha\ll 1$, the approximation is in good agreement with the inverse Lévy transform solution [Eq. (31)].

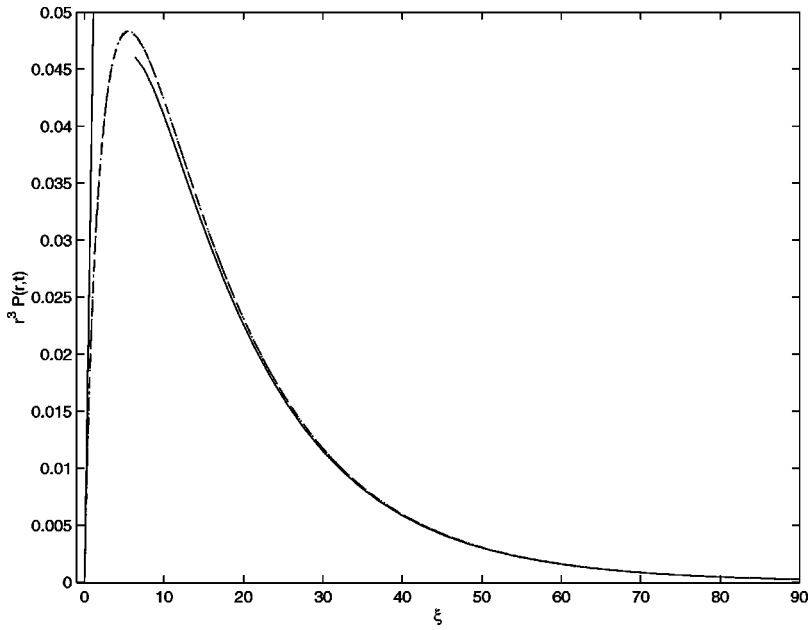

FIG. 3. Scaling behavior of $P(r,t)$ for $\alpha=1/2$ and $d=3$. We show $r^3P(r,t)$ vs $\xi=r^2/t^\alpha$ for times $t=0.02$ (dotted line), $t=2$ (dash dotted line), and $t=200$ (dashed line). Due to the exact scaling of the solution one cannot distinguish between the three curves in the figure. Also shown (solid curves) are the two asymptotic behaviors valid for small and large values of the scaling variable ξ. For $\xi\gg 1$ we used Eq. (34), and for $\xi\ll 1$ we used the leading term in the expansion [Eq. (35)].

$$\hat{P}(x,u)\sim\begin{cases}\dfrac{F}{T}u^{\alpha-1}\tau^{\alpha}\exp\left[-\dfrac{F}{T}\tau^{\alpha}u^{\alpha}x\right] & x>0\\ \dfrac{F}{T}u^{\alpha-1}\tau^{\alpha}\exp\left[-\dfrac{F}{T}|x|(1+\tau^{\alpha}u^{\alpha})\right] & x\leqslant 0.\end{cases} \tag{40}$$

Since $\int_{-\infty}^{0}\hat{P}(x,u)\sim u^{\alpha-1}$ and, according to a Tauberian theorem, $\int_{-\infty}^{0}P(x,t)\sim 1/t^{\alpha}$, hence $P(x,t)=0$ for $x<0$ when $t\to\infty$. Therefore, using the inverse Laplace transform of Eq. (40) the asymptotic behavior of $P(x,t)$ is

$$\lim_{t\to\infty}P(x,t)=\begin{cases}\dfrac{1}{\alpha A^{1/\alpha}}\dfrac{t}{x^{1+1/\alpha}}l_{\alpha}\left(\dfrac{t}{A^{1/\alpha}x^{1/\alpha}}\right) & 0<x\\ 0 & 0>x,\end{cases} \tag{41}$$

and $A=(F/T)\tau^{\alpha}$. Integrating Eq. (41), we find the distribution function

$$\lim_{t\to\infty}\int_{-\infty}^{x}P(x,t)dx=1-L_{\alpha}\left(\frac{t}{A^{1/\alpha}x^{1/\alpha}}\right), \tag{42}$$

valid for $x>0$. Equation (42) was also derived in Ref. [49], based on the biased CTRW in the limit $t\to\infty$; thus, the solution of the fractional Fokker-Planck equation in a linear external field, converges to the solution of the biased CTRW in the limit of large t.

At the origin one can use the Tauberian theorem to show that

$$P(0,t)\sim\frac{A}{\Gamma(1-\alpha)}t^{-\alpha} \tag{43}$$

valid for long times. For the case $F=0$ we found $P(0,t)\sim t^{-\alpha/2}$, so, as expected, the decay on the origin is faster for the biased case, since particles are drifting away from the origin.

In Fig. 4 we present the solution for the case $\alpha=1/2$; then, according to Eq. (29),

$$P(x,t)=\frac{1}{\sqrt{tK_{1/2}^2\pi}}\int_0^{\infty}\frac{1}{\sqrt{4\pi t'}}\times\exp\left[-\frac{t'^2}{4K_{1/2}^2t}-\frac{(x-Ft'/T)^2}{4t'}\right]dt', \tag{44}$$

which is evaluated numerically. For large times,

$$P(x,t)\sim\frac{A}{\sqrt{\pi t}}\exp\left[-\frac{A^2x^2}{4t}\right] \tag{45}$$

for $0<x$. As seen in the figure, the exact result exhibits a strong sensitivity on the initial condition, and the maximum of $P(x,t)$ is located on $x=0$. This is different from an ordinary diffusion process, in which the maximum of $P(x,t)$ is on $\langle x(t)\rangle$. The curves in Fig. 4 are similar to those observed by Scher and Montroll [21], based on lattice simulation of the CTRW and also by Weissman *et al.* [50] who investigated the biased CTRW using an analytical approach. The FFPE solution presented here is much simpler than the CTRW solution, still it captures all the important features of the more complex CTRW result.

C. Example 2: fractional first passage time problem for the fractional biased Wiener process

The first passage time is the time a particle first leaves a given domain. We shall consider the first passage time of a one-dimensional particle in the presence of an external linear field $F(x)=F>0$. The investigation of the first passage time problem in a nonlinear force field will soon follow. At time $t=0$ the particle is located on $x=0$, and the absorbing boundary is on $x=a>0$, which mathematically means $P(a,t)=0$. Solutions of the FFPE with special boundary

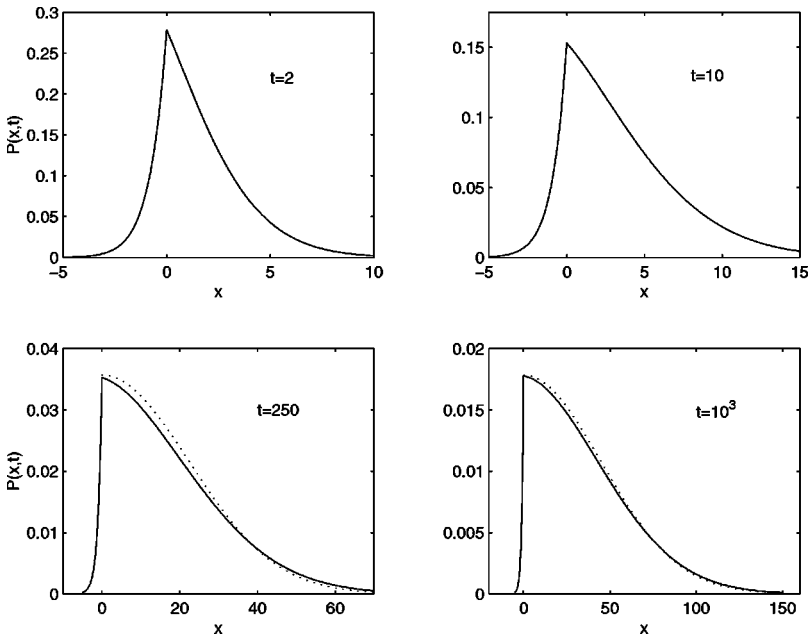

FIG. 4. The dynamics of $P(x,t)$ for the fractional diffusion process in an external linear force field F. We show the solution for four different times. The dotted curve is the asymptotic solution valid for large t. The maximum of $P(x,t)$ is for all times on the initial condition $x=0$. We use $F=T=K_{1/2}=1$ and $\alpha=1/2$.

conditions can be found using the standard methods of solution of the ordinary Fokker-Planck equation, and we use the method of images to find, in Laplace space,

$$\hat{P}(x,u)=\hat{\xi}(x,u)-e^{Fa/T}\hat{\xi}(x-2a,u), \quad -\infty<x<a, \tag{46}$$

where

$$\hat{\xi}(x,u)=\frac{Fu^{\alpha-1}\tau^{\alpha}}{T\sqrt{1+4(u\tau)^{\alpha}}}\exp\left[\frac{F(x-\sqrt{1+4u^{\alpha}\tau^{\alpha}}|x|)}{2T}\right] \tag{47}$$

is the solution of the FFPE with the free boundary condition [Eq. (39)]. Since we showed in Sec. III B how to obtain $\xi(x,t)$, one can also find $P(x,t)$.

The survival probability $S(t)$ (i.e., the probability that the particle did not reach a until time t) in Laplace u space is

$$\hat{S}(u)=\int_{-\infty}^{a}\hat{P}(x,u)dx. \tag{48}$$

Inserting Eq. (46) into Eq. (48), and integrating, we find

$$\hat{S}(u)=\frac{1}{u}\left\{1-\exp\left[\frac{F(1-\sqrt{1+4u^{\alpha}\tau^{\alpha}})a}{2T}\right]\right\}. \tag{49}$$

Let t_f be the random time it takes the particle to reach a for the first time. The probability density function of the first passage time t_f is

$$\eta(t_f)=-\frac{dS(t_f)}{dt_f}, \tag{50}$$

or, in Laplace $t_f\rightarrow u$ space,

$$\hat{\eta}(u)=-u\hat{S}(u)+1. \tag{51}$$

Using Eq. (49), we find

$$\hat{\eta}(u)=\exp\left[\frac{F(1-\sqrt{1+4u^{\alpha}\tau^{\alpha}})a}{2T}\right]. \tag{52}$$

Since $\hat{\eta}(u=0)=1$, $\eta(t_f)$ is a normalized PDF. For $F=0$ our result reduces to what was found previously by Balakrishnan [6]:

$$\hat{\eta}(u)=\exp\left(-\frac{u^{\alpha/2}}{\sqrt{K_{\alpha}}}a\right); \tag{53}$$

thus for $F=0$, $\eta(t_f)$ is a one sided Lévy stable density with index $\alpha/2$, and therefore, for large t_f

$$\eta(t_f)\sim\frac{a\alpha}{2\Gamma(1-\alpha/2)\sqrt{K_{\alpha}}}t_f^{-(1+\alpha/2)}. \tag{54}$$

For $\alpha=1$ and $F=0$, we find the well known solution of the integer first passage time [51]. For finite F, we find the large time behavior of $\eta(t_f)$ using the small u behavior of Eq. (52). A short calculation yields

$$\eta(t_f)\sim\frac{aT\alpha}{\Gamma(1-\alpha)FK_{\alpha}}t_f^{-(1+\alpha)}, \tag{55}$$

valid for $\alpha<1$. This behavior is very different from the standard case when $F>0$ and $\alpha=1$, which exhibits an exponential decay. The decay found in Eq. (55) for $F>0$ is faster than the decay found in Eq. (54) for $F=0$; this is expected, since the external field transports the particle toward the absorbing boundary. In Fig. 5 we show the survival probability $S(t)$ for $\alpha=1/2$. To find $S(t)$ we first find $\xi(x,t)$, the solution of the FFPE with a free boundary condition [Eq. (44)]; we then find $P(x,t)$ defined in Eq. (46), and integrate over x according to Eq. (48). For short times $S(t)=1$, since then the probability packet does not reach the absorbing boundary at

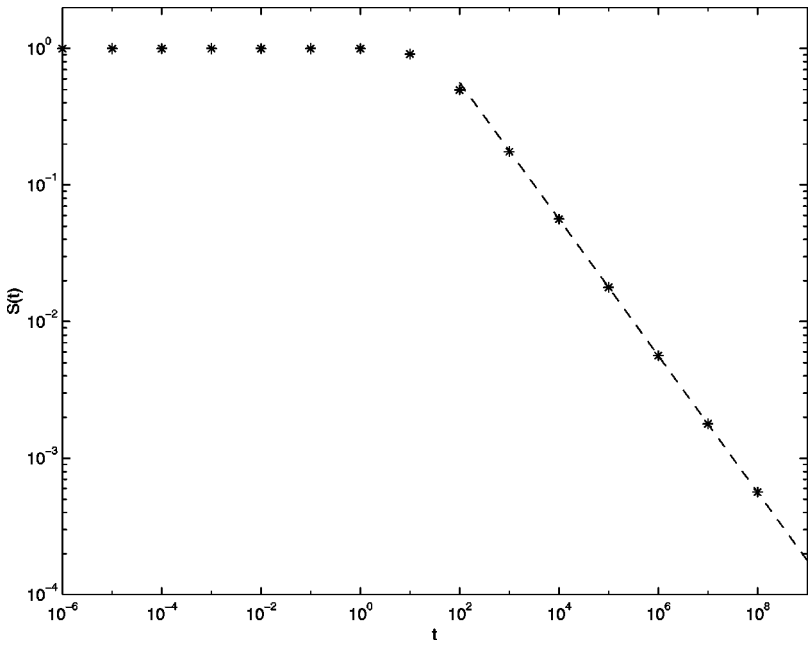

FIG. 5. The survival probability (stars) for the biased fractional Wiener process vs t. We choose $\alpha=1/2$, $K_{1/2}=F=T=1$, and $a=10$. The dashed curve is the asymptotic behavior $S(t)\sim t^{-1/2}/\sqrt{\pi}$.

a. After a transient time the survival probability follows $S(t)\sim t^{-\alpha}$, as predicted in Eqs. (50) and (55). The transient time will be investigated further in Sec. V in the context of dispersive time of flight experiments.

D. Example 3: fractional first passage time problem in a nonlinear field

Let us consider the more general case of a one-dimensional particle in a force field $F(x)$ with initial condition $\delta(x-x_0)$ and absorbing boundary condition $P(a,t)=0$, where $a>x_0$. We show that the inverse Lévy transform can be used to find the relation between the fractional survival probability $S_\alpha(t)$ and the corresponding $S_1(t)$. Using Eq. (29) and $S_\alpha(t)=\int_{-\infty}^{a}P(x,t)dx$, changing the order of integrations we find

$$S_\alpha(t)=\frac{1}{\alpha}\left(\frac{K_\alpha}{K_1}\right)^{1/\alpha}t\int_0^\infty\frac{1}{t'^{1+1/\alpha}}l_\alpha\left(\frac{K_\alpha^{1/\alpha}t}{K_1^{1/\alpha}t'}\right)S_1(t')dt', \tag{56}$$

and, in Laplace space,

$$\hat{S}_\alpha(u)=\left(\frac{K_1}{K_\alpha}\right)u^{\alpha-1}\hat{S}_1\left(\frac{K_1}{K_\alpha}u^\alpha\right). \tag{57}$$

From this relation it can be shown that, if $S_1(t)$ is a sum of purely decaying exponentials, $S_\alpha(t)$ is a sum of purely decaying Mittag-Leffler functions with index α [i.e., this is similar to behavior in Eqs. (26)–(28)]. The PDF of escape times $\eta(t_f)$ is, in Laplace u space,

$$\hat{\eta}_\alpha(u)=-u\hat{S}_\alpha(u)+1; \tag{58}$$

therefore using

$$\hat{\eta}_\alpha(u)=-u^\alpha\frac{K_1}{K_\alpha}\int_0^\infty e^{-u^\alpha(K_1/K_\alpha)t}S_1(t)dt+1, \tag{59}$$

$$S_1(t)=-\int_0^t\eta_1(t')dt'+1, \tag{60}$$

and the convolution theorem of Laplace transform, we find

$$\hat{\eta}_\alpha(u)=\hat{\eta}_1\left(u^\alpha\frac{K_1}{K_\alpha}\right). \tag{61}$$

It is easy to verify that Eqs. (52) and (53) are special cases of Eq. (61).

Since $\hat{\eta}_\alpha(u=0)=\hat{\eta}_1(u=0)$, normalization of the possibly defected (i.e., un-normalized) PDF $\eta_\alpha(t_f)$ is identical to the normalization of the corresponding $\eta_1(t_f)$. $\eta_1(t_f)$ is not normalized if there exist a finite probability of not reaching the boundary in the time interval $(0,\infty)$. For example, when the force field is negative and linear, $F(x)=-F<0$ with $x_0<a$, since then the net transport occurs in a direction away from the boundary and not all particles reach point a. For such a defected case it might be more convenient to consider the survival probability $S(t)$, not $\eta(t_f)$.

E. Example 4: first passage time in three dimensions

Our results in Sec. III D can be generalized to higher dimensions. Assume that a $d=3$ fractional random walk is described by a force free fractional Fokker-Planck equation, with the initial condition

$$P(\mathbf{r},t=0)=\frac{\delta(r-R_0)}{4\pi r^2}. \tag{62}$$

Let the random walk terminate once the particle reaches a sphere with radius $a<R_0$, whose center is at the origin. The solution of the first passage time, from R_0 to a, is found using either the method of images or the inverse Lévy transform. We find

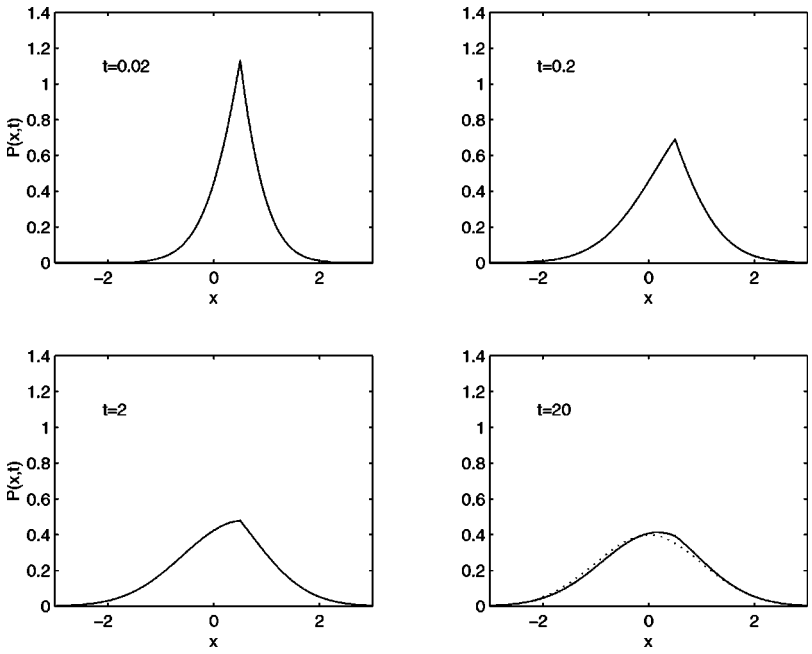

FIG. 6. The dynamics of $P(x,t)$ for the fractional Ornstein-Uhlenbeck process with $\alpha=1/2$ and $K_{1/2}=1$. We show the solution for four different times. The dotted curve is the stationary solution, i.e., the Boltzmann distribution which for the harmonic potential is Gaussian. Note the cusp on the initial condition $x=x_0=1/2$.

$$\hat{\eta}(u)=\frac{a}{R_0}\exp\left[-(R_0-a)\frac{u^{\alpha/2}}{K_\alpha^{1/2}}\right]; \tag{63}$$

hence $(R_0/a)\,\eta(t_f)$ is a one sided Lévy stable density with index $\alpha/2$. Note that Eq. (63) is similar to the corresponding one-dimensional solution [Eq. (53)]. Solution (63) is not normalized:

$$\int_0^\infty \eta(t_f)dt_f=\frac{a}{R_0}<1. \tag{64}$$

This type of behavior might be expected based on Polya's problem. It is well known that for three-dimensional ordinary random walks on a lattice the probability of return to the origin is not 1. Here we find a finite survival probability $S(t=\infty)=1-a/R_0$, which is α independent.

F. Example 5, fractional Ornstein-Uhlenbeck process

We consider the fractional Ornstein–Uhlenbeck (OU) process, namely, the motion of a test particle in a harmonic potential investigated previously in Ref. [18]. This case cannot be analyzed using the CTRW in a direct way since the CTRW formalism considers only uniformly biased random walks. We consider $\alpha=1/2$ and $F(x)/T=-x$, and use the well known solution of the ordinary OU process [25,26]:

$$P_1(x,s)=\frac{1}{\sqrt{2\pi(1-e^{-2s})}}\exp\left[-\frac{(x-x_0e^{-s})^2}{2(1-e^{-2s})}\right]. \tag{65}$$

The solution of the fractional OU process is then found using a numerical integration of Eq. (14) using Eqs. (A7) and (65). Our results are presented in Fig. 6. We have considered an initial condition $x_0=1/2$, and we observe a strong dependence of the solution on the initial condition. A cusp at $x=x_0$ is observed for all times t; thus the initial condition has a strong influence on the solution. The solution approaches the stationary Gaussian shape slowly, in a power law way, and the solution deviates from Gaussian for any finite time. Unlike the ordinary Gaussian OU process, the maximum of $P(x,t)$ is not on the average $\langle x(t)\rangle$; rather the maximum for short times is located at the initial condition.

Like the ordinary OU process, the fractional OU process has a special role. The ordinary OU process describes two types of behaviors: the first is an overdamped motion of a particle in a harmonic potential, the second is the velocity of a Brownian particle modeled by the Langevin equation; the latter is the basis for the Kramers equation. In a similar way the fractional OU process describes an overdamped and anomalous motion in the harmonic potential considered in this section and in Ref. [12]; it can also be used to model the velocity of a particle exhibiting a Lévy walk type of motion [18]. The fractional OU process is the basis of the fractional Kramers equation introduced recently by Barkai and Silbey [18]. This equation describes super-diffusion, while in this work we consider subdiffusion.

IV. CONTINUOUS TIME RANDOM WALK

We now discuss the meaning of the inverse Lévy transform based on the CTRW. The decoupled CTRW, in dimension d, describes a process for which a particle is trapped at the origin for time t_1, then jumps to $\mathbf{r_1}$, is trapped on $\mathbf{r_1}$ for time t_2, and then jumps to a new location; the process is then renewed. Let $\psi(t)$ be the PDF of the independent identically distributed (IID) pausing times between successive steps, and $f(r)$ the PDF of the IID displacements. In what follows we assume $f(r)$ has a finite variance and a zero mean. The asymptotic behavior of the CTRW is well investigated [10,16,49,52–55].

Let $P(r,t)$ be the PDF of finding the CTRW particle at r at time t. Let $N_{\mathrm{CT}}(s,t)$ be the probability that s steps are made in the time interval $(0,t)$, and the subscript CT denotes

the CTRW. Because the model is decoupled,

$$P(r,t)=\sum_{s=0}^{\infty} N_{\mathrm{CT}}(s,t)W(r,s), \tag{66}$$

and $W(r,s)$ is the probability density that the particle has reached r after s steps. $W(r,s)$ will generally depend on $f(r)$; however, it is usually assumed that in the large time limit only contributions from large s are important. From the standard central limit theorem, we know

$$W(r,s)\rightarrow_{s\rightarrow\infty} G(r,s)=\frac{1}{(4\pi s)^{d/2}}\exp\left(-\frac{r^2}{4s}\right), \tag{67}$$

where we use convenient units. Because the steps are independent, the convolution theorem of the Laplace transform yields

$$\hat{N}_{\mathrm{CT}}(s,u)=\frac{1-\hat{\psi}(u)}{u}\exp\{s\ln[\hat{\psi}(u)]\}, \tag{68}$$

and $\hat{N}_{\mathrm{CT}}(s,u)$ is the Laplace transform of $N_{\mathrm{CT}}(s,t)$. If, for $u\rightarrow 0$, $\hat{\psi}(u)\sim 1-u^{\alpha}\cdots$ [i.e., $\psi(t)\sim t^{-(1+\alpha)}$], then, in the small u limit corresponding to large t,

$$\hat{N}_{\mathrm{CT}}(s,u)\sim u^{\alpha-1}\exp(-su^{\alpha}). \tag{69}$$

The CTRW $\hat{N}_{\mathrm{CT}}(s,u)$ is identical to $\hat{n}(s,u)$ found within the context of the FFPE. Replacing the summation in Eq. (66) with an integration, using Eqs. (67) and (69) and the inverse Laplace transform, one finds

$$P(r,t)\sim\int_0^{\infty} n(s,t)G(r,s)ds. \tag{70}$$

According to Eq. (70), derived previously in Ref. [15], the large time behavior of the CTRW is described by the inverse Lévy transform of the Gaussian $G(r,s)$. Thus, when $F(x)=0$, the FFPE describes the long time behavior of the CTRW, and when $F(x)\neq 0$ it describes a CTRW type of behavior in a force field.

Remark 1. If $f(r)$ is broad (i.e., the variance of jump length diverges) the fractional Riemann-Liouville approach is not valid; instead an approach based on fractional space derivatives is appropriate [17].

Remark 2. From Eq. (70) we learn that the inverse Lévy transform has a meaning of a generalized law of large numbers. When $\alpha=1$, the mean pausing time is finite; therefore, the law of large numbers is valid, and we expect that the number of steps follows $s\sim t$. Indeed, for this normal case, $n(s,t)=\delta(s-t)$. When $\alpha<1$ the law of large numbers is not valid; instead the random number of steps s is described by the PDF $n(s,t)$.

Remark 3. For $r=0$ and $d>1$, Eq. (70) is not valid. To see this, consider, for simplicity, $W(r,s)=G(r,s)$ and $d=3$. Then, using $N_{CT}(s,t)\leqslant 1$ and Eq. (66),

$$\lim_{r\rightarrow 0}P(r,t)\leqslant N_{CT}(0,t)\delta(\mathbf{r})+\sum_{s=1}^{\infty}\frac{1}{\sqrt{4\pi s}^{d}}, \tag{71}$$

while, according to Eq. (35), $P(r,t)\sim 1/r$ when $r\rightarrow 0$. It is easy to see that the $1/r$ behavior found within the solution of the fractional diffusion equation is due to the $s\rightarrow 0$ behavior of $n(s,t)$. This shortcoming found within the framework of the fractional kinetic equation is due to the fact that we give statistical weight to CTRW's with $s<1$ steps (i.e., replace summation with integration).

CTRW, an exact solution

The derivation of Eq. (70) was not mathematically rigorous, and our aim now is to check whether it works well for a specific example. We find an exact solution of the CTRW, expressed in terms of an infinite sum of known functions, and compare between the exact result and the asymptotic expression [Eq. (70)].

The solution of the CTRW in $\mathbf{k},u$ space is [10]

$$\hat{P}(\mathbf{k},u)=\frac{1-\hat{\psi}(u)}{u}\frac{1}{1-\hat{\psi}(u)f(\mathbf{k})}. \tag{72}$$

Usually the CTRW solution is found based on a numerical inverse Fourier-Laplace transform of Eq. (72), or using the Monte Carlo approach. Here we find an exact solution of the CTRW process for a special choice of $f(r)$ and $\psi(t)$. We assume the PDF of jump times $\psi(t)$ to be a one sided Lévy stable density, with $\hat{\psi}(u)=\exp(-u^{\alpha})$. Displacements are assumed to be Gaussian, and then

$$W(r,s)=\frac{1}{(4\pi s)^{d/2}}\exp\left(-\frac{r^2}{4s}\right) \tag{73}$$

is exact, not only asymptotic. For this choice of PDF's the solution of the CTRW can be found explicitly. We use

$$\hat{N}_{\mathrm{CT}}(s,u)=\frac{1-\exp(-u^{\alpha})}{u}\exp(-su^{\alpha}), \tag{74}$$

and the convolution theorem of Laplace transform to find

$$N_{\mathrm{CT}}(0,t)=1-L_{\alpha}(t),$$

$$N_{\mathrm{CT}}(s,t)=L_{\alpha}\left(\frac{t}{s^{1/\alpha}}\right)-L_{\alpha}\left(\frac{t}{(s+1)^{1/\alpha}}\right), \tag{75}$$

and

$$L_{\alpha}(t)=\int_0^t l_{\alpha}(t)dt \tag{76}$$

is the one sided Lévy stable distribution. Inserting Eqs. (73) and (75) into Eq. (66), we find

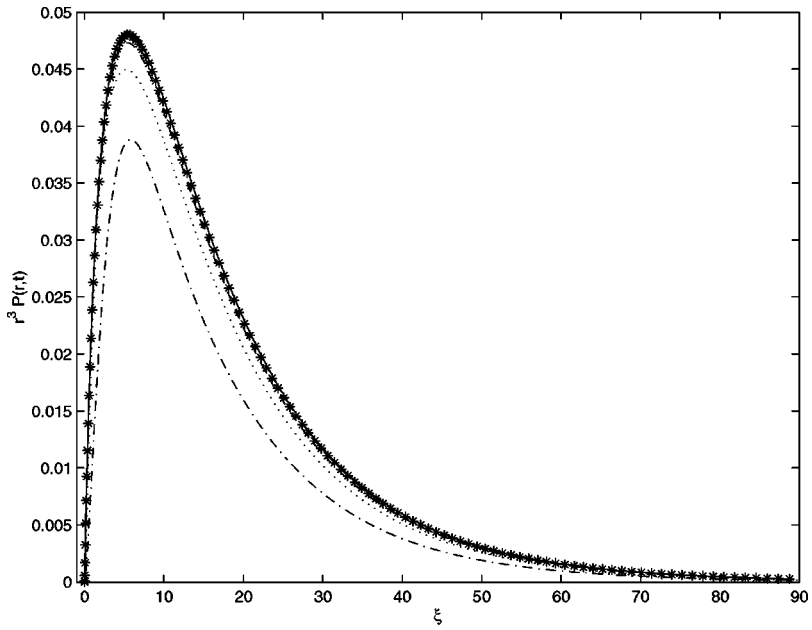

FIG. 7. We show $r^3P(r,t)$ vs $\xi=r^2/t^\alpha$ for the CTRW process and for $\alpha=1/2$ $d=3$. The curves in the figure, $t=2$ (dot dashed line), $t=20$ (dotted line), $t=200$ (dashed line), and $t=2000$ (stars) converge in the limit $t\to\infty$ to the asymptotic curve (solid). The asymptotic curve was computed using Eq. (70). It is difficult to distinguish between the results for $t=200$ and 2000 and the asymptotic result. Not shown is the delta function contribution at $r=0$.

$$P(r,t)=[1-L_\alpha(t)]\delta(r)+\sum_{s=1}^{\infty}\left\{L_\alpha\left[\frac{t}{s^{1/\alpha}}\right]-L_\alpha\left[\frac{t}{(s+1)^{1/\alpha}}\right]\right\}\frac{1}{(4\pi s)^{d/2}}\exp\left(-\frac{r^2}{4s}\right). \quad (77)$$

The first term on the right hand side describes random walks for which the particle did not leave the origin within the observation time t; the other terms describe random walks where the number of steps is s. In Fig. 7 we show the solution of the CTRW process in a scaling form. We consider a three-dimensional case, $\alpha=1/2$, and use

$$L_{1/2}(t)=\frac{1}{\sqrt{\pi}}\Gamma[1/2,1/(4t)]. \quad (78)$$

Here $\Gamma(\cdot,\cdot)$ denotes the incomplete gamma function. The figure shows $r^3P(r,t)$ versus the scaling variable $\xi=r^2/t^\alpha$. For large times t the CTRW solution converges to the solution of the fractional diffusion equation already presented in Fig. 3.

Let us calculate the Cartesian moments of the CTRW process,

$$M(2m_1,\ldots,2m_d)=\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_d x_1^{2m_1}\cdots x_d^{2m_d}P(x_1,\ldots,x_d,t), \quad (79)$$

with non-negative integers $m_1,m_2,\ldots$. Clearly, the odd moments equal zero, and, from normalization, $M(0,\ldots,0)=1$. In Appendix C we find

$$M(2m_1,\ldots,2m_d)=C_{m,d}\sum_{s=1}^{\infty}\left\{L_\alpha\left[\frac{t}{s^{1/\alpha}}\right]-L_\alpha\left[\frac{t}{(s+1)^{1/\alpha}}\right]\right\}s^m, \quad (80)$$

with

$$C_{m,d}=\frac{2^{2m}}{\pi^{d/2}}\prod_{i=1}^{d}\Gamma\left(m_i+\frac{1}{2}\right), \quad (81)$$

and $m=\Sigma_i^d m_i>0$. In the Appendixes we also show that, for $t\to\infty$,

$$M(2m_1,\ldots,2m_d)\sim C_{m,d}\Gamma(1+m)\frac{t^{\alpha m}}{\Gamma(1+\alpha m)}. \quad (82)$$

To derive Eq. (82), we used the small u expansion of the Laplace transform of Eq. (80), and the Tauberian theorem. One can easily show that the moments in Eq. (82) are identical to the moments obtained directly from the integral transformation [Eq. (70)]. Hence our interpretation of Eq. (70) as the asymptotic solution of the CTRW is justified [however, our derivation of Eq. (82) is based on a specific choice of $\psi(t)$ and $f(r)$].

V. SCHER-MONTROLL TRANSPORT

In this section we show that the FFPE approach is compatible with the SM model for dispersive transport. Scher and Montroll used a CTRW with an absorbing boundary condition to model photoconductivity in amorphous semiconductor As_2Se_3 and an organic compound TNF-PVK, finding $\alpha\simeq0.5$ and $\alpha=0.8$, respectively. In the semiconductor experiment, holes are injected near a positive electrode, and then transported to a negative electrode where they are

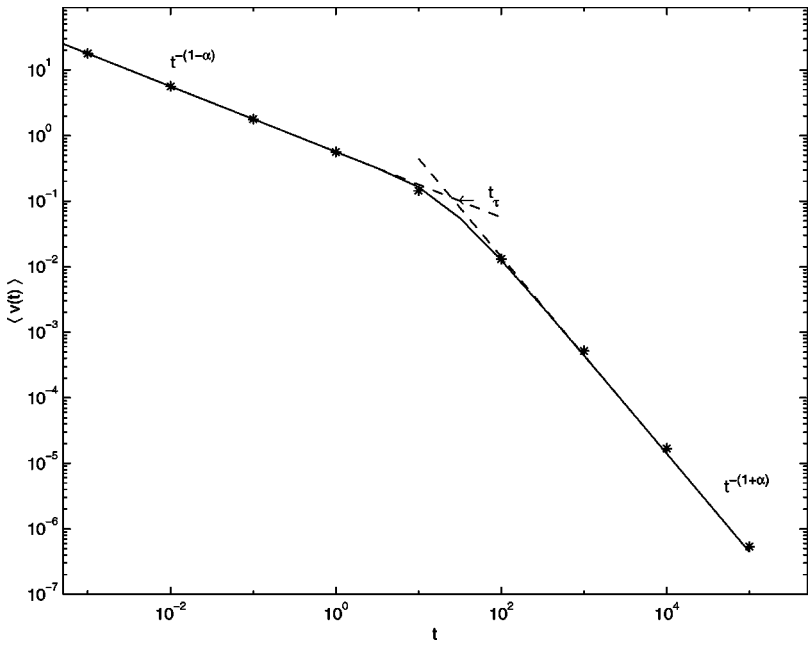

FIG. 8. We show the velocity vs time. For short (long) times we observe $t^{-(1-\alpha)}$ and $[t^{-(1+\alpha)}]$ decays which characterize current in time of flight experiments in disordered medium. The dashed curves are the asymptotic expressions given in Eq. (86). The solid curve is found using Eq. (93) and the stars are based on exact numerical solution of the FFPE. Also indicated is the transition time t_τ. We use $\alpha=1/2$, $K_{1/2}=F=T=1$, and $a=10$.

absorbed. The experiments showed that the current is not compatible with Gaussian transport. The transient current in these time of flight experiments follows two types of behaviors,

$$I(t)\sim\begin{cases} t^{-(1-\alpha)} & t<t_\tau \\ t^{-(1+\alpha)} & t>t_\tau, \end{cases} \tag{83}$$

where t_τ is a transient time. For Gaussian transport processes charge carriers move at a constant velocity, and after a transient time, depending on the thickness of the sample and external field, the charge carrier is absorbed. Hence, for normal processes found in most ordered material, the current has a step shape. Since experimental data of normalized current $I(t)/I(t_\tau)$ versus time t/t_τ, for different sample thickness and external fields, collapse onto one scaling curve described by Eq. (83), the phenomena was designated universal. The widespread occurrence of these features in a variety of disordered materials confirms the universality of the approach [22–24,56,57]

SM showed how to calculate $\psi(t)$ using an effective medium approach, and averaging over many exponential processes. The exponent α may depend on temperature and on other control parameters. Here we consider α as a fit parameter. The basic assumptions SM used are as follows:

(i) The current of charge carriers is modeled by one dimensional process. Since $\langle x(t)\rangle\neq 0$ only in the direction of the applied external linear field, this assumption works well.

(ii) The absorbing boundary condition at $x=a$ and the effect of the other boundary conditions are neglected. Since the transport is in the direction of the absorbing boundary, this assumption is reasonable.

(iii) The measured current is given by $I(t)\propto\langle v(t)\rangle$. $\langle v(t)\rangle$ is the time derivative of the mean displacement $\langle x(t)\rangle$, which is calculated based on the decoupled CTRW on a lattice.

(iv) Additional assumptions are discussed in the literature [21,58–60].

We describe the transport process based on the FFPE,

$$\frac{\partial P(x,t)}{\partial t}={}_0D_t^{1-\alpha}K_\alpha\left(-\frac{\partial}{\partial x}\frac{F}{T}+\frac{\partial^2}{\partial x^2}\right)P(x,t), \tag{84}$$

with initial condition $P(x,t=0)=\delta(x)$ and the absorbing boundary condition $P(a,t)=0$. Due to the boundary, condition $P(x,t)$ is not normalized. Therefore, the meaning of $\langle x(t)\rangle$ is not straightforward. We imagine that a charge carrier, once reaching the boundary at a, is trapped there and not removed. Therefore,

$$\langle x(t)\rangle=\int_{-\infty}^{a}xP(x,t)dx+[1-S(t)]a, \tag{85}$$

where $S(t)$ is the survival probability. From Eq. (85), $\langle x(t=0)\rangle=0$ and $\langle x(t=\infty)\rangle=a$. The first term on the right hand side of Eq. (85) describes processes in the bulk, while the second describes the reduction of the number of particles participating in the process.

Since in Sec. IV we showed how to calculate $P(x,t)$ and $S(t)$, using the inverse Lévy transform, the calculation of $\langle x(t)\rangle$ and hence of $\langle v(t)\rangle$ becomes straightforward. The integrals involved are easily calculated using standard numerical packages, and we used MATHEMATICA. Our results for the averaged velocity and $\alpha=1/2$ are presented in Fig. 8.

The time behavior of $\langle x(t)\rangle$ is investigated in Appendix D, based on the small u behavior of $\hat{P}(x,u)$ and $\hat{S}(u)$. When $a\gg T/F$, we find

$$\langle v(t)\rangle\sim\begin{cases} \dfrac{\alpha FK_\alpha}{T\Gamma(1+\alpha)}t^{-(1-\alpha)} & t\ll t_r \\ \dfrac{\alpha a^2T}{2FK_\alpha\Gamma(1-\alpha)}t^{-(1+\alpha)} & t\gg t_\tau, \end{cases} \tag{86}$$

which is similar to Eq. (83). These behaviors can be easily understood. For short times the absorbing boundary has no effect, and $\langle v(t)\rangle$ is the time derivative of Eq. (37). For long times the second term on the right hand side of Eq. (85) is important, using Eqs. (50) and (55):

$$\langle v(t)\rangle \sim -a\dot{S}(t) = a\,\eta(t) \sim t^{-1-\alpha}. \tag{87}$$

As shown in Appendix D, the bulk term in Eq. (85) [i.e., $\int_{-\infty}^{a} xP(x,t)dx$] also contributes to the current, this contribution being negative.

A convenient definition of t_τ is the time at which the extrapolated asymptotic dashed lines in Fig. 8 intersect. Using Eq. (86),

$$t_\tau = \left[\frac{\Gamma(1+\alpha)}{2\Gamma(1-\alpha)}\right]^{1/2\alpha}\left(\frac{aT}{FK_\alpha}\right)^{1/\alpha}. \tag{88}$$

Since t_τ is measurable, F, T, and a are known system parameters, and α can be determined from the current slopes, one can determine the value of K_α from experiment. If we assume K_α is independent of the force field F, we find

$$t_\tau \sim \left(\frac{a}{F}\right)^{1/\alpha}, \tag{89}$$

in agreement with the SM prediction.

Let us now further demonstrate the effectiveness of the inverse Lévy transform. The mean velocity in an ideal Gaussian transport system (i.e., $\alpha=1$) is given by a step function behavior

$$\langle v_1(t)\rangle = \frac{FK_1}{T}, \quad t<(aT/FK_1), \tag{90}$$

and $\langle v_1(t)\rangle = 0$ otherwise. This behavior is a simplification of the approximately step current observed in time of flight experiments in normal (i.e., ordered) systems. The mean displacement in such an ideal process is

$$\langle x_1(t)\rangle = \begin{cases} \dfrac{FK_1 t}{T} & t<(aT/FK_1) \\ a & t>(aT/FK_1). \end{cases} \tag{91}$$

The velocity for the fractional process is found in two steps. First find $\langle x(t)\rangle$ by applying the inverse Lévy transform [Eq. (29)], to the mean displacement of ideal Gaussian process [Eq. (91)]. Second, take the time derivative of the mean displacement to find $\langle v(t)\rangle$. The result we find is

$$\langle v(t)\rangle = \frac{\alpha FK_\alpha}{T} t^{\alpha-1}\int_{t^*}^{\infty} dz\,\frac{l_\alpha(z)}{z^\alpha}. \tag{92}$$

$t^* = [FK_\alpha/(Ta)]^{1/\alpha} t$ is a dimensionless time in the problem. For short times t^* we may take the lower limit of the integral to zero; using $\int_0^\infty dz\, l_\alpha(z)/z^\alpha = 1/\Gamma(1+\alpha)$, we find the short time behavior in Eq. (86). The long time behavior in Eq. (86) is also easily found by noting that $l_\alpha(z) \sim \alpha z^{-(1+\alpha)}/\Gamma(1-\alpha)$ for $z\to\infty$. For $\alpha = 1/2$, we find

$$\langle v(t)\rangle = \frac{1}{\sqrt{\pi}}\frac{FK_{1/2}t^{-1/2}}{T}\left[1-\exp\left(-\frac{T^2a^2}{4F^2K_{1/2}^2 t}\right)\right]; \tag{93}$$

as shown in Fig. 8, this equation interpolates between the short and long time behaviors of $\langle v(t)\rangle$.

Using Eqs. (88) and (92), we prove the scaling behavior of the current, similar to what is observed in many experiments. We find

$$\frac{I(t)}{I(t_\tau)} = \frac{\langle v(t)\rangle}{\langle v(t_\tau)\rangle} = F\left(\frac{t}{t_\tau}\right), \tag{94}$$

and the scaling function is

$$S(x) = c_2 x^{\alpha-1}\int_{c_1 x}^{\infty}\frac{l_\alpha(z)}{z^\alpha}dz, \tag{95}$$

with

$$c_2 = \left[\int_{c_1}^{\infty}\frac{l_\alpha(z)}{z^\alpha}dz\right]^{-1}, \tag{96}$$

and $c_1 = [\Gamma(1+\alpha)/(2\Gamma(1-\alpha))]^{1/\alpha}$. We see that the inverse Lévy transform approach predicts the scaling function $S(x)$ for the time of flight experiments. Within the CTRW we expect the scaling function to depend on details of $\psi(t)$ in the vicinity of the transition time t_τ.

The FFPE is compatible with linear response theory, and the generalized Einstein relations hold [13,30] (i.e., assuming α and K_α are independent of the external field). Hence the FFPE describes a physical system in a weak external field. Experiments are not limited to weak external fields, and then one might be able to fit data using a field dependent K_α. Barkai and Fleurov [30] calculated $\psi(t)$ for a specific type of disorder and in an external field, and predicted a transition to Gaussian behavior for long enough times and strong enough fields. Thus we expect the FFPE to give the correct behavior only in the linear response regime, or, when the field becomes strong, only within a certain time span.

VI. SUMMARY

The fractional Fokker-Planck equation (FFPE) is a simple single exponent stochastic framework describing subdiffusive transport in a nonlinear external field and close to thermal equilibrium. The inverse Lévy transform gives the solution of the FFPE in terms of the corresponding solution of an ordinary Fokker-Planck equation. The integral transformation also describes the long time behavior of the CTRW in dimensions $d=1$, 2, and 3. Thus the transformation maps Gaussian diffusion onto fractional diffusion, and it can serve as a tool for finding the solution of certain fractional kinetic equations.

The solution of the CTRW for a special choice of $f(r)$ and $\psi(t)$ was also found. This solution is a useful tool for a detailed comparison between the CTRW and the FFPE. The exact solution can also be used to check the accuracy of numerical solutions of the CTRW.

All through this work we have emphasized the relation between the FFPE and stable laws. The basis of the FFPE is the generalized central limit applied to the number of steps in the corresponding random walk. In addition, for a particle in a binding external potential the stationary solution is the Boltzmann equilibrium. These features, and the relation of the FFPE with the CTRW, are the foundations of the FFPE. Recently, a fractional Kramers equation was introduced [18]. This equation has similar features to the FFPE; however, it describes superdiffusion.

Transport in ordered media is often modeled using the diffusion equation, this approach being the simplest and most widely used. Dispersive Scher-Montroll transit time type experiments, observed in a large number of disordered systems, can be described phenomenologically using the FFPE. This is only one example of physical phenomena in which a different kind of calculus, i.e., noninteger calculus, plays a central role.

Note added After this work was completed a review article on the FFPE [61], and work on the fractional first passage time problem [62,63], were published. This work focused on subdiffusion. In Ref. [64], an extension of the solution method discussed in this paper for the fractional wave equation (superdiffusion) was considered.

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APPENDIX A

Lévy stable PDF's are usually defined by means of their Fourier transform. One sided stable PDF's with $l_\alpha(z)=0$ for $z<0$ and $0<\alpha<1$ are defined by

$$\int_{-\infty}^{\infty} e^{ikz} l_\alpha(z)dz = \exp(-|k|^\alpha e^{-i(k/|k|)(\pi/2)\alpha}). \quad \text{(A1)}$$

It is convenient to express one sided stable PDF's in terms of the Laplace transform

$$\int_0^\infty e^{-uz} l_\alpha(z)dz = e^{-u^\alpha}, \quad u>0. \quad \text{(A2)}$$

Thus $l_\alpha(z)$ can be found using the (e.g., numerical) inverse Fourier or inverse Laplace transform. Schneider [65] represented Lévy stable densities in terms of Fox the H function [46,47]:

$$l_\alpha(z)=\frac{1}{\alpha z^2} H_{1,1}^{1,0}\left[z^{-1} \middle| \begin{matrix} (-1,1) \\ (-1/\alpha, 1/\alpha) \end{matrix} \right]. \quad \text{(A3)}$$

The asymptotic small z behavior yields [65]

$$l_\alpha(z) \sim B z^{-\sigma} e^{-\kappa z^{-\tau}}, \quad \text{(A4)}$$

where

$$\tau=\frac{\alpha}{(1-\alpha)}, \quad \kappa=(1-\alpha)\alpha^{\alpha/(1-\alpha)}, \quad \sigma=\frac{2-\alpha}{2(1-\alpha)}$$

$$B=\{[2\pi(1-\alpha)]^{-1}\alpha^{1/(1-\alpha)}\}^{1/2}. \quad \text{(A5)}$$

The large z series expansion is

$$l_\alpha(z)=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Gamma(1+n\alpha)}{n!}(-1)^{n-1}\sin(\pi n\alpha)z^{-(1+n\alpha)}, \quad \text{(A6)}$$

and, using the well known formula, the $n=1$ term yields $l_\alpha(z)\sim\alpha\Gamma(1-\alpha)^{-1}z^{-(1+\alpha)}$.

Representation of the one sided stable PDF's in terms of other special functions is usually obtained by examining expansion (A6) and using properties of Γ functions. It was shown [21] that, for any rational value of α, $l_\alpha(z)$ can be expressed in terms of a finite sum of hypergeometric functions [e.g., $l_{3/4}(z)$ given in Ref. [21]]. We list other one sided stable densities in terms of more familiar functions.

(i) The one sided PDF with $\alpha=1/2$, also known as Smirnov's density,

$$l_{1/2}(z)=\frac{1}{2\sqrt{\pi}}z^{-3/2}e^{-1/(4z)}. \quad \text{(A7)}$$

This result can be easily verified using the Laplace transform.

(ii) The one sided stable PDF with $\alpha=2/3$ is given in terms of Whittaker's W function

$$l_{2/3}(z)=\sqrt{\frac{3}{\pi}}z^{-1}e^{-y/2}W_{1/2,1/6}(y), \quad \text{(A8)}$$

where $y=4/(27z^2)$. Note that for this case, [65] points relevant errors in the literature.

(iii) For $\alpha=1/3$, in terms of the modified Bessel function of the second kind

$$l_{1/3}(z)=\frac{1}{3\pi}z^{-3/2}K_{1/3}\left(\frac{2}{\sqrt{27z}}\right). \quad \text{(A9)}$$

We use the expansion in Eq. (A6) and the MATHEMATICA command Simplify[$\cdots$], to find for $\alpha=1/4$

$$l_{1/4}(z)=(6\sqrt{2}\sqrt{z}\Gamma(5/4)\mathrm{HPFQ}[\{\},\{1/2,3/4\},-1/(256z)] -3\sqrt{\pi}z^{1/4}\mathrm{HPFQ}[\{\},\{3/4,5/4\},-1/(256z)] +\sqrt{2}\Gamma(7/4)\mathrm{HPFQ}[\{\},\{5/4.3/2\}, -1/(256z)])/(12\pi z^{7/4}), \tag{A10}$$

where HPFQ[$\cdots$]=HypergeometricPFQ[$\cdots$] is the hypergeometric function and we are using MATHEMATICA notation. Other representations of stable densities $l_\alpha(z)$, for rational exponents α, can be found in similar way. However, we used MATHEMATICA Version 4 to find the properties of Eq. (A10) for $0<z\ll 1$, and found negative values for $l_{1/4}(z)$, which were wrong. This probably implies some type of numerical problem within MATHEMATICA, at least for $\alpha=1/4$.

APPENDIX B

In this appendix we find the small $\xi\ll 1$ series expansion of $P(r,t)$ based on the Fox H function solution. The Fox function is represented as

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1)\cdots(a_p,\alpha_p)\\(b_1,\beta_1)\cdots(b_q,\beta_q)\end{matrix}\right.\right], \tag{B1}$$

and, for our choice of parameters,

$$m=2,\quad n=0,\quad p=1,\quad q=2,$$

$$a_1=1,\quad \alpha_1=1,$$

$$b_1=d/2,\quad \beta_1=1/\alpha,\quad b_2=1,\quad \beta_2=1/\alpha. \tag{B2}$$

The asymptotic expansion of the H Fox function, for $0<x\ll 1$, is defined when two conditions are satisfied [46,47]. The first is

$$\delta=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j>0, \tag{B3}$$

and for the case $\delta=0$, see Refs. [46,47]. For our case, defined by the parameters in Eq. (B2), $\delta=2/\alpha-1>0$ when $0<\alpha<1$. The second condition is

$$\beta_h(b_j+\Lambda)\neq\beta_j(b_h+k) \tag{B4}$$

for

$$j\neq h,\quad j=h=1,\ldots m,\quad \Lambda,k=0,1,2,\ldots. \tag{B5}$$

Using Eq. (B2) condition (B5) reads

$$\frac{1}{\alpha}(1+\Lambda)\neq\frac{1}{\alpha}(d/2+k); \tag{B6}$$

therefore, the condition is satisfied for dimensions $d=1$ and 3, but not for $d=2$:

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1)\cdots(a_p,\alpha_p)\\(b_1,\beta_1)\cdots(b_q,\beta_q)\end{matrix}\right.\right] =\sum_{h=1}^{m}\sum_{k=0}^{\infty}\Pi_{j=1,j\neq h}^{m}\Gamma(b_j-\beta_j\xi_{h,k})\Pi_{j=1}^{n}\Gamma(1-a_j +\alpha_j\xi_{h,k})(-1)^k(x)^{\xi_{h,k}}[\Pi_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi_{h,k}) \times\Pi_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi_{h,k})k!\beta_h]^{-1}, \tag{B7}$$

where

$$\xi_{h,k}=(b_h+k)/\beta_h. \tag{B8}$$

Using Eq. (B7), we find

$$H_{1,2}^{2,0}\left[x\left|\begin{matrix}(1,1)\\(d/2,1/\alpha),(1,1/\alpha)\end{matrix}\right.\right] =\alpha\left\{\sum_{k=0}^{\infty}\frac{\Gamma(1-d/2-k)(-1)^k x^{\alpha(d/2+k)}}{\Gamma(1-\alpha d/2-\alpha k)k!} +\sum_{k=0}^{\infty}\frac{\Gamma(d/2-1-k)(-1)^k x^{\alpha(1+k)}}{\Gamma(1-\alpha-\alpha k)k!}\right\}. \tag{B9}$$

Using some simple manipulations, for $d=1$ we find

$$P(r,t)=\frac{1}{2t^{\alpha/2}}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{n/2}}{n!\Gamma[1-\alpha(n+1)/2]} \tag{B10}$$

and, for $d=3$, Eq. (35).

For $d=2$ the asymptotic expansion of the Fox function is not known. Using Eq. (31) one can show that for $d=2$, and $\xi\ll 1$

$$P(r,t)\sim\frac{1}{\pi\Gamma(1-\alpha)t^{\alpha}}\ln[t^{\alpha/2}/r]. \tag{B11}$$

Equation (B11) was derived independently by Saichev [66].

APPENDIX C

We investigate the Cartesian moments of the CTRW. These moments are

$$M(2m_1,\ldots,2m_d)=\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_d x_1^{2m_1}x_2^{m_2}\cdots x_d^{2m_d} \times\sum_{s=1}^{\infty}\left\{L_\alpha\left[\frac{t}{s^{1/\alpha}}\right] -L_\alpha\left[\frac{t}{(s+1)^{1/\alpha}}\right]\right\}\frac{1}{(4\pi s)^{d/2}} \times\exp\left(-\frac{x_1^2+\cdots+x_d^2}{4s}\right). \tag{C1}$$

Changing the order of integration and summation, we use the identity

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_d \frac{1}{(4\pi s)^{d/2}} \times \exp\left(-\frac{x_1^2+\cdots+x_d^2}{4s}\right) x_1^{2m_1}\cdots x_d^{2m_d} = C_{m,d} s^m \tag{C2}$$

where $C_{m,d}$ is defined in Eq. (81). Inserting Eq. (C2) in Eq. (C1) we find Eq. (80).

The Laplace transform of Eq. (80) is

$$M(2m_1,\ldots,2m_d) = C_{m,d}\frac{1-e^{-u^\alpha}}{u}\sum_{s=1}^{\infty} e^{-su^\alpha} s^m. \tag{C3}$$

We use

$$\sum_{s=1}^{\infty} e^{-su^\alpha} s^m = (-1)^m \left(\frac{d}{dx}\right)^m \sum_{s=1}^{\infty} e^{-xs}\Big|_{x=u^\alpha} = (-1)^m \left(\frac{d}{dx}\right)^m \frac{e^{-x}}{1-e^{-x}}\Big|_{x=u^\alpha}, \tag{C4}$$

and, for small u^α, we find

$$M(2m_1,\ldots,2m_d) \sim C_{m,d}\Gamma(m+1)u^{-1-\alpha m}. \tag{C5}$$

Applying Tauberian theorem [i.e., inverting Eq. (C5)] we find Eq. (82).

APPENDIX D

In Laplace space the mean displacement is written as a sum of three terms,

$$\langle \hat{x}(u)\rangle = \langle \hat{x}_1(u)\rangle + \langle \hat{x}_2(u)\rangle + \langle \hat{x}_3(u)\rangle, \tag{D1}$$

with

$$\langle \hat{x}_1(u)\rangle \equiv \int_{-\infty}^{a} x\hat{\xi}(x,u)dx, \tag{D2}$$

$$\langle \hat{x}_2(u)\rangle \equiv e^{Fa/T}\int_{-\infty}^{a} x\hat{\xi}(x-2a,u)dx, \tag{D3}$$

and

$$\langle \hat{x}_3(u)\rangle \equiv \left[\frac{1}{u}-\hat{S}(u)\right]a. \tag{D4}$$

We define

$$E \equiv \frac{Fu^{\alpha-1}\tau^\alpha}{TG}, \tag{D5}$$

$$G \equiv \sqrt{1+4u^\alpha\tau^\alpha}, \tag{D6}$$

$$B_\pm \equiv \frac{F}{2T}(1\pm G). \tag{D7}$$

Then it is easy to show that

$$\langle \hat{x}_1(u)\rangle = E\frac{(aB_- -1)e^{B_- a}+1}{B_-^2} - \frac{E}{B_+^2}. \tag{D8}$$

For $u^\alpha\tau^\alpha \to 0$, we find

$$\langle \hat{x}_1(u)\rangle \sim \left(\frac{a^2F}{2T}-\frac{T}{F}\right)u^{\alpha-1}\tau^\alpha. \tag{D9}$$

We use the condition $a \gg T/F$, and then

$$\langle \hat{x}_1(u)\rangle \sim \frac{a^2F}{2T}u^{\alpha-1}\tau^\alpha. \tag{D10}$$

We therefore find, for long times,

$$\langle x_1(t)\rangle \sim \frac{a^2T}{2FK_\alpha}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{D11}$$

Using the same approach,

$$\langle \hat{x}_2(u)\rangle = -Ee^{Fa/T}e^{-B_+ a}\frac{1+aB_+}{B_+^2}. \tag{D12}$$

For $(u\tau)^\alpha \to 0$, we find

$$\langle \hat{x}_2(u)\rangle = -u^{\alpha-1}\tau^\alpha a, \tag{D13}$$

and, therefore, for $t\to\infty$,

$$\langle x_2(t)\rangle = -\frac{aT^2}{FK_\alpha}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{D14}$$

When $a \gg T/F$, $\langle x_1(t)\rangle \gg \langle x_2(t)\rangle$ for $t\to\infty$. Using Eq. (50), it is easy to see that

$$\langle \dot{x}_3(t)\rangle = \eta(t)a \tag{D15}$$

and $\eta(t)$ is the first passage time PDF. Using Eqs. (55), (D11), and (D14), we find

$$\langle v(t)\rangle \sim \frac{\alpha a^2 T}{2FK_\alpha\Gamma(1-\alpha)}t^{-(1+\alpha)}, \tag{D16}$$

valid for $t\to\infty$ and when $a \gg T/F$. We note that both $\langle \dot{x}_1(t)\rangle$ and $\langle \dot{x}_3(t)\rangle$ contribute to $\langle v(t)\rangle$, the contribution from $\langle \dot{x}_1(t)\rangle$ being negative.

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